GEOMETRIC CRYSTALS ON UNIPOTENT GROUPS AND GENERALIZED YOUNG TABLEAUX

NAKASHIMA Toshiki *
Department of Mathematics,
Sophia University, Tokyo 102-8554, Japan
e-mail: toshiki@mm.sophia.ac.jp

Abstract

We define geometric/unipotent crystal structure on unipotent subgroups of semi-simple algebraic groups. We shall show that in A_n-case, their ultra-discretizations coincide with crystals obtained by generalizing Young tableaux.

Key words: Geometric crystal, Unipotent groups, Generalized Young tableaux

1 Introduction

The notion of crystals is initiated by Kashiwara ([3],[4],[5]), which influences over many areas in mathematics, in particular, combinatorics and representation theory, *e.g.,* combinatorics of Young tableaux(= semi-standard tableaux), piece-wise linear combinatorics, *etc.* Indeed, in [7], we succeed in describing the crystal bases for classical quantum algebras by using Young tableaux. One feature of crystal theory is that it produces many piece-wise linear formulae ([5],[11],[12],[13]).

Theory of geometric crystals is introduced by Berenstein and Kazhdan [1] in semi-simple setting and is extended to Kac-Moody setting in [10], which is a kind of geometric analogue of Kashiwara's crystal theory. More precisely, let G be a Kac-Moody group over $\mathbb{C}$, T be its maximal torus and I be a finite index set of its simple roots. For an ind-(algebraic)variety X, morphisms $e_i : \mathbb{C}^\times \times X \to X$ $(i \in I)$ and $\gamma : X \to T$, the triplet $(X, \gamma, \{e_i\}_{i\in I})$ is called a *geometric crystal* if they satisfy the conditions as in Definition 2.2. Geometric crystals are not only analogy of crystals, but also has certain categorical correpondence to crystals, which is called a tropicalization/ultra-discretization. It is so remarkable that this correpondence reproduces several piece-wise linear formulae in the theory of crystals from subtraction free(=positive) rational formulae in geometric crystals ([10]) as follows:

$$\begin{array}{ccc} \{\text{Geometric Crystals}\} & \underset{\text{tropicalization}}{\overset{\text{ultra-discretization}}{\rightleftarrows}} & \{\text{Crystals}\} \\ x \times y,\ x/y,\ x+y & & x+y,\ x-y,\ \max(x,y) \end{array}$$

Furthermore, this correpondence reproduces the tensor product structure of crystals from the product structure of geometric crystals ([1]).

Let B be a Borel subgroup of G and W be the Weyl group associated with G. Any finite Schubert variety $\overline{X}_w \subset X := G/B$ has a natural geometric crystal structure([1],[10]). Then, in semi-simple setting we know that the whole flag variety $X := G/B$ holds a geometric crystal structure. But, in general Kac-Moody setting, we do not have any natural geometric crystal structure on the flag variety X. The opposite unipotent subgroup U^- can be seen as

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an open dense subset of X. In this paper, we present some sufficient condition for existence of geometric(unipotent) crystal structure on U^- and then on X, which is described as follows: if there exists a morphism $\mathcal{T} : U^- \to T$ satisfying the condition as in Lemma 3.2, then we obtain U-morphism $F : U^- \to B^-$ and then the associated unipotent crystal structure, which means the exsistence of a geometric crystal structure on U^-. In semi-simple cases, there exists such morphism which is given by matrix coefficients. In particular, for $G = SL_{n+1}(\mathbb{C})$ case, we present its geometric crystal structure explicitly and reveal that it corresponds to the crystals called *generalized Young tableaux*, which is a sort of "limit" of usual Young tableaux and forms a free $\mathbb{Z}$-lattice of rank $\frac{n(n+1)}{2}$. In more general cases, *e.g.,* affine cases, the existence of such morphisms is not yet known, which is our further problem.

The article is organized as follows: in Sect.2, we review the notion of geometric crystals, unipotent crystals and the tropicalization/ultra-discretization correspondence. In Sect.3, we consider geometric crystal on a unipotent subgroup $U^- \subset G$ and in Sect.4, the explicit geometric crystal strcutre on $U^- \subset SL_{n+1}(\mathbb{C})$ is described. In the final section, we give a tropcialization/ultra-discretization correpondence between geometric crystals on U^- and generalized Young tableaux.

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2 Geometric Crystals and Unipotent Crystals

2.1 Kac-Moody algebras and Kac-Moody groups

Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$, where I be a finite index set. Let $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{h_i\}_{i\in I})$ be the associated root data, where $\mathfrak{t}$ be the vector space over $\mathbb{C}$ with dimension $|I|+$ corank(A), and the set of simple roots $\{\alpha_i\}_{i\in I} \subset \mathfrak{t}^*$ and the set of simple co-roots $\{h_i\}_{i\in I} \subset \mathfrak{t}$ are linearly independent indexed sets satisfying $\alpha_i(h_j) = a_{ij}$.

The Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with A is the Lie algebera over $\mathbb{C}$ generated by $\mathfrak{t}$, the Chevalley generators e_i and f_i $(i \in I)$ with the usual defining relations ([8],[9]). There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{t}^*} \mathfrak{g}_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in \mathfrak{t}^* | \alpha \neq 0,\ \mathfrak{g}_\alpha \neq (0)\}$. Set $Q := \sum_i \mathbb{Z}\alpha_i$, $Q_+ := \sum_i \mathbb{Z}_{\geq 0}\alpha_i$ and $\Delta_+ := \Delta \cap Q_+$. An element of Δ_+ is called a positive root. Let ω be the Chevalley involution of $\mathfrak{g}$ defined by $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$ and $\omega(h) = -h$ for $h \in \mathfrak{t}$. Let $L(\Lambda)$ ($\Lambda \in P_+$: set of dominant weights) be an irreducible integrable highest weight module with highest weight Λ and $\pi_\Lambda : \mathfrak{g} \to \mathrm{End}(L(\Lambda))$ be the $\mathfrak{g}$-action. The action $\pi^*_\Lambda := \pi_\Lambda \circ \omega$ defines a $\mathfrak{g}$-module structure on $L(\Lambda)$, which is called the contragredient module of $L(\Lambda)$ and denoted $L^*(\Lambda)$. Let us fix a highest weight vector $u_\Lambda \in L(\lambda)$ and denote it by u^*_Λ in $L^*(\Lambda)$. We obtain a unique $\mathfrak{g}$-invariant bilinear form $\langle \quad , \quad \rangle$ on $L(\Lambda) \times L^*(\Lambda)$ such that $\langle u_\Lambda, u^*_\Lambda \rangle = 1$.

Define simple reflections $s_i \in \mathrm{Aut}(\mathfrak{t})$ $(i \in I)$ by $s_i(h) := h - \alpha_i(h)h_i$, which generate the Weyl group W. We also define the action of W on $\mathfrak{t}^*$ by $s_i(\lambda) := \lambda - \alpha(h_i)\alpha_i$. Set $\Delta^{\mathrm{re}} := \{w(\alpha_i) | w \in W,\ i \in I\}$, whose element is called a real root.

Let G be the Kac-Moody group associated with the derived Lie algebra $\mathfrak{g}'$ defined in [9]. Set $U_\alpha := \exp \mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$, which is an one-parameter subgroup of G and G is generated by U_α $(\alpha \in \Delta^{\mathrm{re}})$. Let $U^\pm$ be the subgroups generated by $U_{\pm\alpha}$ $(\alpha \in \Delta^{\mathrm{re}}_+ = \Delta^{\mathrm{re}} \cap Q_+)$, *i.e.,* $U^\pm := \langle U_{\pm\alpha} | \alpha \in \Delta^{\mathrm{re}}_+ \rangle$, which is called the unipotent subgroup of G. Here note that if $\mathfrak{g}$ is a semi-simple Lie algebra, then G is a usual semi-simple algebraic group over $\mathbb{C}$.

For any $i \in I$, there exists a unique homomorphism; $\phi_i : SL_2(\mathbb{C}) \to G$ such that

$$x_i(t) := \phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \exp te_i,\ y_i(t) := \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp tf_i \quad (t \in \mathbb{C}).$$

Set $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \phi_i(\{\mathrm{diag}(t, t^{-1}) | t \in \mathbb{C}\})$ and $N_i := N_{G_i}(T_i)$. Let T (resp. N) be the subgroup of G generated by T_i (resp. N_i), which is called a *maximal torus* in G

and $B^{\pm} := U^{\pm}T$ be the Borel subgroup of G. We have the isomorphism $\phi : W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_iT/T$. An element $\overline{s}_i := x_i(-1)y_i(1)x_i(-1)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$. Define $R(w)$ for $w \in W$ by

$$R(w) := \{(i_1, i_2, \cdots, i_l) \in I^l | w = s_{i_1} s_{i_2} \cdots s_{i_l}\},$$

where l is the length of w. We associate to each $w \in W$ its standard representative $\bar{w} \in N_G(T)$ by $\bar{w} = \bar{s}_{i_1}\bar{s}_{i_2} \cdots \bar{s}_{i_l}$ for any $(i_1, i_2, \cdots, i_l) \in R(w)$.

We have the following (as for ind-variety and ind-group, see [6]):

Proposition 2.1 ([6]). (*i*) *Let G be a Kac-Moody group and $U^{\pm}$, $B^{\pm}$ be its subgroups as above. Then G is an ind-group and $U^{\pm}$, $B^{\pm}$ are its closed ind-subgroups.*

(*ii*) *The multiplication maps*

$$\begin{array}{ccccccc} T \times U & \longrightarrow & B & \qquad & U^- \times T & \longrightarrow & B^- \\ (t, u) & \mapsto & tu & & (v, t) & \mapsto & vt \end{array}$$

are isomorphisms of ind-varieties.

2.2 Geometric Crystals

In this subsection, we review the notion of geometric crystals ([1],[10]).

Let $(a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix and G be the associated Kac-Moody group with the maximal torus T. An element in $\mathrm{Hom}(T, \mathbb{C}^{\times})$ (resp. $\mathrm{Hom}(\mathbb{C}^{\times}, T)$) is called a *character* (resp. *co-character*) of T. We define a *simple co-root* $\alpha_i^{\vee} \in \mathrm{Hom}(\mathbb{C}^{\times}, T)$ $(i \in I)$ by $\alpha_i^{\vee}(t) := T_i$. We have a pairing $\langle \alpha_i^{\vee}, \alpha_j \rangle = a_{ij}$.

Let X be an ind-variety over $\mathbb{C}$, $\gamma : X \to T$ be a rational morphism and a family of rational morphisms $e_i : \mathbb{C}^{\times} \times X \to X$ $(i \in I)$;

$$\begin{array}{rccc} e_i^c & : \mathbb{C}^{\times} \times X & \longrightarrow & X \\ & (c, x) & \mapsto & e_i^c(x). \end{array}$$

For a word $\mathbf{i} = (i_1, i_2, \cdots, i_l) \in R(w)$ $(w \in W)$, set $\alpha^{(l)} := \alpha_{i_l}$, $\alpha^{(l-1)} := s_{i_l}(\alpha_{i_{l-1}})$, $\cdots$, $\alpha^{(1)} := s_{i_l} \cdots s_{i_2}(\alpha_{i_1})$. Now for a word $\mathbf{i} = (i_1, i_2, \cdots, i_l) \in R(w)$ we define a rational morphism $e_{\mathbf{i}} : T \times X \to X$ by

$$(t, x) \mapsto e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)} e_{i_2}^{\alpha^{(2)}(t)} \cdots e_{i_l}^{\alpha^{(l)}(t)}(x).$$

Definition 2.2. (i) The triplet $\chi = (X, \gamma, \{e_i\}_{i \in I})$ is a *geometric crystal* if it satisfies $e^1(x) = x$ and

$$\gamma(e_i^c(x)) = \alpha_i^{\vee}(c)\gamma(x), \tag{2.1}$$

$$e_{\mathbf{i}} = e_{\mathbf{i}'} \quad \text{for any } w \in W, \text{ and any } \mathbf{i}, \mathbf{i}' \in R(w). \tag{2.2}$$

(ii) Let $(X, \gamma_X, \{e_i^X\}_{i \in I})$ and $(Y, \gamma_Y, \{e_i^Y\}_{i \in I})$ be geometric crystals. A rational morphism $f : X \to Y$ is a *morphism of geometric crystals* if f satisfies that

$$f \circ e_i^X = e_i^Y \circ f, \quad \gamma_X = \gamma_Y \circ f.$$

In particular, if a morphism f is a birational isomorphism of ind-varieties, it is called an *isomorphism of geometric crystals*.

The following lemma is a direct result from [1][Lemma 2.1] and the fact that the Weyl group of any Kac-Moody Lie algebra is a Coxeter group [2][Proposition 3.13].

Lemma 2.3. *The relations (2.2) are equivalent to the following relations:*

$$\begin{array}{ll}
e_i^{c_1}e_j^{c_2}=e_j^{c_2}e_i^{c_1} & \text{if } \langle\alpha_i^\vee,\alpha_j\rangle=0,\\
e_i^{c_1}e_j^{c_1c_2}e_i^{c_2}=e_j^{c_2}e_i^{c_1c_2}e_j^{c_1} & \text{if } \langle\alpha_i^\vee,\alpha_j\rangle=\langle\alpha_j^\vee,\alpha_i\rangle=-1,\\
e_i^{c_1}e_j^{c_1^2c_2}e_i^{c_1c_2}e_j^{c_2}=e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^2c_2}e_i^{c_1} & \text{if } \langle\alpha_i^\vee,\alpha_j\rangle=-2,\ \langle\alpha_j^\vee,\alpha_i\rangle=-1,\\
e_i^{c_1}e_j^{c_1^2c_2}e_i^{c_1^3c_2}e_j^{c_1^3c_2^2}e_i^{c_1c_2}e_j^{c_2}=e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^3c_2^2}e_i^{c_1^3c_2}e_j^{c_1^2c_2}e_i^{c_1} & \text{if } \langle\alpha_i^\vee,\alpha_j\rangle=-3,\ \langle\alpha_j^\vee,\alpha_i\rangle=-1,
\end{array}$$

Remark. If $\langle\alpha_i^\vee,\alpha_j\rangle\langle\alpha_j^\vee,\alpha_i\rangle\geq 4$, there is no relation between e_i and e_j.

2.3 Unipotent Crystals

In the sequel, we denote the unipotent subgroup U^+ by U. We define unipotent crystals (see [1]) associated to Kac-Moody groups. The definitions below follow [1],[10].

Definition 2.4. Let X be an ind-variety over $\mathbb{C}$ and $\alpha : U\times X\to X$ be a rational U-action such that α is defined on $\{e\}\times X$. Then, the pair $\mathbf{X}=(X,\alpha)$ is called a *U-variety.* For U-varieties $\mathbf{X}=(X,\alpha_X)$ and $\mathbf{Y}=(Y,\alpha_Y)$, a rational morphism $f : X\to Y$ is called a *U-morphism* if it commutes with the action of U.

Now, we define the U-variety structure on $B^-=U^-T$. By Proposition 2.1, B^- is an ind-subgroup of G and then is an ind-variety over $\mathbb{C}$. The multiplication map in G induces the open embedding; $B^-\times U\hookrightarrow G$, then this is a birational isomorphism. Let us denote the inverse birational isomorphism by g;

$$g : G\longrightarrow B^-\times U.$$

Then we define the rational morphisms $\pi^- : G\to B^-$ and $\pi : G\to U$ by $\pi^- := \mathrm{proj}_{B^-}\circ g$ and $\pi := \mathrm{proj}_U\circ g$. Now we define the rational U-action α_{B^-} on B^- by

$$\alpha_{B^-} := \pi^-\circ m : U\times B^-\longrightarrow B^-,$$

where m is the multiplication map in G. Then we obtain U-variety $\mathbf{B}^-=(B^-,\alpha_{B^-})$.

Definition 2.5. (i) Let $\mathbf{X}=(X,\alpha)$ be a U-variety and $f : X\to\mathbf{B}^-$ be a U-morphism. The pair $(\mathbf{X},f)$ is called a *unipotent G-crystal* or, for short, *unipotent crystal.*

(ii) Let $(\mathbf{X},f_X)$ and $(\mathbf{Y},f_Y)$ be unipotent crystals. A U-morphism $g : X\to Y$ is called a *morphism of unipotent crystals* if $f_X=f_Y\circ g$. In particular, if g is a birational isomorphism of ind-varieties, it is called an *isomorphism of unipotent crystals.*

We define a product of unipotent crystals following [1]. For unipotent crystals $(\mathbf{X},f_X)$, $(\mathbf{Y},f_Y)$, define a morphism $\alpha_{X\times Y} : U\times X\times Y\to X\times Y$ by

$$\alpha_{X\times Y}(u,x,y) := (\alpha_X(u,x),\alpha_Y(\pi(u\cdot f_X(x)),y)). \tag{2.3}$$

If there is no confusion, we use abbreviated notation $u(x,y)$ for $\alpha_{X\times Y}(u,x,y)$.

Theorem 2.6 ([1]). (*i*) *The morphism $\alpha_{X\times Y}$ defined above is a rational U-morphism on $X\times Y$.*

(*ii*) *Let $\mathbf{m} : B^-\times B^-\to B^-$ be a multiplication morphism and $f=f_{X\times Y} : X\times Y\to B^-$ be the rational morphism defined by*

$$f_{X\times Y} := \mathbf{m}\circ(f_X\times f_Y).$$

Then $f_{X\times Y}$ is a U-morphism and then, $(\mathbf{X}\times\mathbf{Y},f_{X\times Y})$ is a unipotent crystal, which we call a product of unipotent crystals $(\mathbf{X},f_X)$ and $(\mathbf{Y},f_Y)$.

(*iii*) *Product of unipotent crystals is associative.*

2.4 From unipotent crystals to geometric crystals

We have the canonical projection $\xi_i : U^- \to U_{-\alpha_i}$ $(i \in I)$ (see [10]). Now, we define the function on U^- by

$$\chi_i := y_i^{-1} \circ \xi_i : U^- \longrightarrow U_{-\alpha_i} \longrightarrow \mathbb{C},$$

and extend this to the function on B^- by $\chi_i(u \cdot t) := \chi_i(u)$ for $u \in U^-$ and $t \in T$. For a unipotent G-crystal $(\mathbf{X}, \mathbf{f_X})$, we define a function $\varphi_i := \varphi_i^X : X \to \mathbb{C}$ by

$$\varphi_i := \chi_i \circ \mathbf{f_X},$$

and a rational morphism $\gamma_X : X \to T$ by

$$\gamma_X := \mathrm{proj}_T \circ \mathbf{f_X} : X \to B^- \to T, \tag{2.4}$$

where proj_T is the canonical projection. Suppose that the function φ_i is not identically zero on X. We define a rational morphism $e_i : \mathbb{C}^\times \times X \to X$ by

$$e_i^c(x) := x_i\left(\frac{c-1}{\varphi_i(x)}\right)(x). \tag{2.5}$$

Theorem 2.7 ([1]). *For a unipotent G-crystal $(\mathbf{X}, \mathbf{f_X})$, suppose that the function φ_i is not identically zero for any $i \in I$. Then the rational morphisms $\gamma_X : X \to T$ and $e_i : \mathbb{C}^\times \times X \to X$ as above define a geometric G-crystal $(X, \gamma_X, \{e_i\}_{i \in I})$, which is called the induced geometric G-crystals by unipotent G-crystal $(\mathbf{X}, f_X)$.*

Due to the product structure of unipotent crystals, we can deduce a product structure of geometric crystals derived from unipotent crystals, which is a counterpart of tensor product structure of Kashiwara's crystals. We omit the explicit statement here (see [1],[10]).

2.5 Crystals

The notion "crystal" is introduced as a combinatorial object by abstracting the properties of "crystal bases", which has, in general, no corresponding $U_q(\mathfrak{g})$-module.

Definition 2.8. A *crystal* B is a set endowed with the following maps:

$$\begin{aligned}
&wt : B \longrightarrow P,\\
&\varepsilon_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\} \quad \text{for} \quad i \in I,\\
&\tilde{e}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\} \quad \text{for} \quad i \in I,\\
&\tilde{e}_i(0) = \tilde{f}_i(0) = 0,
\end{aligned}$$

those maps satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle, \tag{2.6}$$

$$wt(\tilde{e}_i b) = wt(b) + \alpha_i \text{ if } \tilde{e}_i b \in B, \tag{2.7}$$

$$wt(\tilde{f}_i b) = wt(b) - \alpha_i \text{ if } \tilde{f}_i b \in B, \tag{2.8}$$

$$\tilde{e}_i b_2 = b_1 \Longleftrightarrow \tilde{f}_i b_1 = b_2 \ (b_1, b_2 \in B), \tag{2.9}$$

$$\varepsilon_i(b) = -\infty \Longrightarrow \tilde{e}_i b = \tilde{f}_i b = 0. \tag{2.10}$$

The operators $\tilde{e}_i$ and $\tilde{f}_i$ are called the *Kashiwara operators*. Indeed, if (L, B) is a crystal base, then B is a crystal.
Remark. A *pre-crystal* is an object satisfying the conditions (2.6)–(2.8).

Let us define $\tilde{s}_i : B \to B$ $(i \in I)$ ([4]) by

$$\tilde{s}_i(b) = \begin{cases} \tilde{e}_i^{-\langle wt(b),h_i\rangle}(b) & \text{if } \langle wt(b), h_i\rangle < 0, \\ \tilde{f}_i^{\langle wt(b),h_i\rangle}(b) & \text{if } \langle wt(b), h_i\rangle \geq 0. \end{cases}$$

Here note that we have $\tilde{s}_i^2 = \mathrm{id}_B$.

Definition 2.9. Let B be a crystal.

(i) If the actions by $\{s_i\}_{i\in I}$ define the action of the Weyl group W on B, we call B a *W-crystal*.

(ii) If $\tilde{e}_i$ or $\tilde{f}_i$ is bijective, then we call B a *free crystal*.

Note that if B is a free crystal, then $\tilde{f}_i = \tilde{e}_i^{-1}$. We frequently denote a free crystal B by $(B, wt, \{\tilde{e}_i\}_{i\in I})$.

2.6 Positive structure and Ultra-discretizations/Tropicalizations

Let us recall the notions of positive structure and ultra-discretization/tropicalization.

The setting below is simpler than the ones in ([1],[10]), since it is sufficient for our purpose. Let $T = (\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) \cong \mathbb{Z}^l$ (resp. $X_*(T) \cong \mathbb{Z}^l$) be the lattice of characters (resp. co-characters) of T. Set $R := \mathbb{C}(c)$ and define

$$\begin{array}{cccc} v: & R \setminus \{0\} & \longrightarrow & \mathbb{Z} \\ & f(c) & \mapsto & \deg(f(c)). \end{array}$$

Here note that for $f_1, f_2 \in R \setminus \{0\}$, we have

$$v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2) \tag{2.11}$$

Let $f = (f_1, \cdots, f_n) : T \to T'$ be a rational morphism between two algebraic tori $T = (\mathbb{C}^\times)^m$ and $T' = (\mathbb{C}^\times)^n$. We define a map $\widehat{f} : X_*(T) \to X_*(T')$ by

$$(\widehat{f}(\xi))(c) := (c^{v(f_1(\xi(c))}, \cdots, c^{v(f_n(\xi(c)))}),$$

where $\xi \in X_*(T)$. Since v satisfies (2.11), the map $\widehat{f}$ is an additive group homomorphism. If we identify $X_*(T)$ (resp. $X_*(T')$)with $\mathbb{Z}^m$ (resp. $\mathbb{Z}^n$) by $\xi(c) = (c^{l_1}, \cdots, c^{l_m}) \leftrightarrow (l_1, \cdots, l_m) \in \mathbb{Z}^m$, we write

$$\widehat{f}(l_1, \cdots, l_m) := (v(f_1(\xi(c))), \cdots, v(f_n(\xi(c)))).$$

A rational function $f(c) \in \mathbb{C}(c)$ $(f \neq 0)$ is *positive* if f can be expressed as a ratio of polynomials with positive coefficients.

Remark. A rational function $f(c) \in \mathbb{C}(c)$ is positive if and only if $f(a) > 0$ for any $a > 0$ (pointed out by M.Kashiwara).

If $f_1,\ f_2 \in R$ are positive, then we have (2.11) and

$$v(f_1 + f_2) = \max(v(f_1), v(f_2)). \tag{2.12}$$

Definition 2.10 ([1]). Let $f = (f_1, \cdots, f_n) : T \to T'$ between two algebraic tori T, T' be a rational morphism as above. It is called *positive*, if the following two conditions are satisfied:

(i) For any co-character $\xi : \mathbb{C}^\times \to T$, the image of ξ is contained in $\mathrm{dom}(f)$.

(ii) For any co-character $\xi : \mathbb{C}^\times \to T$, any $f_i(\xi(c))$ $(i \in I)$ is a positive rational function.

Denote by $\mathrm{Mor}^+(T, T')$ the set of positive rational morphisms from T to T'.

Lemma 2.11 ([1]). *For any positive rational morphisms $f \in \mathrm{Mor}^+(T_1, T_2)$ and $g \in \mathrm{Mor}^+(T_2, T_3)$, the composition $g \circ f$ is in $\mathrm{Mor}^+(T_1, T_3)$.*

By Lemma 2.11, we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational morphisms.

Lemma 2.12 ([1]). *For any algebraic tori T_1, T_2, T_3, and positive rational morphisms $f \in \mathrm{Mor}^+(T_1, T_2)$, $g \in \mathrm{Mor}^+(T_2, T_3)$, we have $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.*

By this lemma, we obtain a functor

$$\begin{array}{cccc} \mathcal{UD}: & \mathcal{T}_+ & \longrightarrow & \mathfrak{Set} \\ & T & \mapsto & X_*(T) \\ & (f : T \to T') & \mapsto & (\widehat{f} : X_*(T) \to X_*(T'))) \end{array}$$

Definition 2.13 ([1]). Let $\chi = (X, \gamma, \{e_i\}_{i \in I})$ be a geometric crystal, T' be an algebraic torus and $\theta : T' \to X$ be a birational isomorphism. The isomorphism θ is called *positive structure* on χ if it satisfies

(i) the rational morphism $\gamma \circ \theta : T' \to T$ is positive.

(ii) For any $i \in I$, the rational morphism $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ defined by $e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $\theta : T \to X$ be a positive structure on a geometric crystal $\chi = (X, \gamma, \{e_i\}_{i \in I}\})$. Applying the functor $\mathcal{UD}$ to positive rational morphisms $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ and $\gamma \circ \theta : T' \to T$ (the notations are as above), we obtain

$$\begin{aligned} \tilde{e}_i &:= \mathcal{UD}(e_{i,\theta}) : \mathbb{Z} \times X_*(T) \to X_*(T) \\ \tilde{\gamma} &:= \mathcal{UD}(\gamma \circ \theta) : X_*(T') \to X_*(T). \end{aligned}$$

Now, for given positive structure $\theta : T' \to X$ on a geometric crystal $\chi = (X, \gamma, \{e_i\}_{i \in I})$, we associate the triplet $(X_*(T'), \tilde{\gamma}, \{\tilde{e}_i\}_{i \in I})$ with a free pre-crystal structure (see [1, 2.2]) and denote it by $\mathcal{UD}_{\theta, T'}(\chi)$. By Lemma 2.3, we have the following theorem:

Theorem 2.14. *For any geometric crystal $\chi = (X, \gamma, \{e_i\}_{i \in I})$ and positive structure $\theta : T' \to X$, the associated pre-crystal $\mathcal{UD}_{\theta, T'}(\chi) = (X_*(T'), \tilde{\gamma}, \{\tilde{e}_i\}_{i \in I})$ is a free W-crystal (see [1, 2.2])*

We call the functor $\mathcal{UD}$ *"ultra-discretization"* instead of "tropicalization" unlike in [1]. And for a crystal B, if there exists a geometric crystal χ, an algebraic torus T in $\mathcal{T}_+$ and a positive structure θ on χ such that $\mathcal{UD}_{\theta, T}(\chi) \cong B$ as crystals, we call χ a *tropicalization* of B.

3 Geometric crystals on unipotent groups

In this section, we associate a geometric/unipotent crystal structure with unipotent subgroup U^- of semi-simple algebraic group G. In particular, for $G = SL_{n+1}(\mathbb{C})$ we describe it explicitly.

3.1 U-variety structure on U^-

In this subsection, suppose that G is a Kac-Moody group as in Sect.2. As mentioned in 2.3, Borel subgroup B^- has a U-variety structure. By the similar manner, we define U-variety structure on U^-. As in 2.3, the multiplication map m in G induces an open embedding; $m : U^- \times B \hookrightarrow G$, then this is a birational isomorphism. Let us denote the inverse birational isomorphism by h;

$$h : G \longrightarrow U^- \times B.$$

Then we define the rational morphisms $\pi^{--} : G \to B^-$ and $\pi^+ : G \to B$ by $\pi^{--} := \mathrm{proj}_{U^-} \circ h$ and $\pi^+ := \mathrm{proj}_B \circ h$. Now we define the rational U-action α_{U^-} on U^- by

$$\alpha_{U^-} := \pi^{--} \circ m : U \times U^- \longrightarrow U^-,$$

Then we obtain

Lemma 3.1. *A pair* $\mathbf{U}^- = (U^-, \alpha_{U^-})$ *is a* U*-variety on a unipotent subgroup* $U^- \subset G$.

3.2 Unipotent/Geometric crystal structure on U^-

In order to define a unipotent crystal structure on U^-, let us construct a U-morphism $F : U^- \to B^-$.

The multiplication map m in G induces an open embedding; $m : U^- \times T \times U \hookrightarrow G$, which is a birational isomorphism. Thus, by the similar way as above, we obtain the rational morphism $\pi^0 : G \to T$. Here note that we have

$$\pi^-(x) = \pi^{--}(x)\pi^0(x) \quad (x \in G). \tag{3.1}$$

Now, we give a sufficient condition for existence of U-morphism F.

Lemma 3.2. *Let* $\mathcal{T} : U^- \to T$ *be a rational morphism satisfying:*

$$\mathcal{T}(\pi^{--}(xu)) = \pi^0(xu)\mathcal{T}(u), \quad \text{for } x \in U \text{ and } u \in U^-. \tag{3.2}$$

Defining a morphism $F : U^- \to B^-$ *by*

$$\begin{array}{cccc} F : & U^- & \longrightarrow & B^- \\ & u & \mapsto & u\mathcal{T}(u), \end{array} \tag{3.3}$$

then the morphism F *is a* U*-morphism* $U^- \to B^-$.

Proof. We may show

$$F(\alpha_{U^-}(x, u)) = \alpha_{B^-}(x, F(u)), \quad \text{for } x \in U \text{ and } u \in U^-. \tag{3.4}$$

As for the left-hand side of (3.4), we have

$$F(\alpha_{U^-}(x, u)) = \pi^{--}(xu)\mathcal{T}(\pi^{--}(xu)) = \pi^{--}(xu)\pi^0(xu)\mathcal{T}(u),$$

where the last equality is due to (3.2). On the other hand, the right-hand side of (3.4) is written by:

$$\alpha_{B^-}(x, F(u)) = \pi^-(xu\mathcal{T}(u)) = \pi^{--}(xu\mathcal{T}(u))\pi^0(xu\mathcal{T}(u)) = \pi^{--}(xu)\pi^0(xu)\mathcal{T}(u)$$

where the second equality is due to (3.1) and the third equality is obtained by the fact that $\mathcal{T}(u) \in T \subset B$. Now we get (3.4). □

Let us verify that there exists such U-morphism F or rational morphism $\mathcal{T}$ for semisimple cases. Suppose that G is semisimple in the rest of this section.

Let $\Lambda_i \in P_+$ $(i = 1, \cdots, n)$ be a fundamental weight and $L(\Lambda_i)$ be a corresponding irreducible highest weight $\mathfrak{g}$-module, where $\mathfrak{g}$ is a complex semi-simple Lie algebra associated with G. Let $L^*(\Lambda_i)$ be a contragredient module of $L(\Lambda_i)$ as in Sect.2 and fix a highest (resp. lowest) weight vector u_{Λ_i} (resp. v_{Λ_i}) and $v^*_{\Lambda_i} \in L^*(\Lambda_i)$ be the same vector as v_{Λ_i} such that $\langle v_{\Lambda_i}, v^*_{\Lambda_i}\rangle = 1$. Now, let us define a function $f_i : U^- \to \mathbb{C}$ $(i \in I)$ as a matrix coefficient:

$$f_i(g) = \langle g \cdot u_{\Lambda_i}, v^*_{\Lambda_i}\rangle. \tag{3.5}$$

We define a rational morphism $\mathcal{T} : U^- \to T$ by

$$\mathcal{T}(u) := \prod_{i\in I} \alpha_i^\vee(f_i(u)^{-1}). \tag{3.6}$$

and define a morphism $F : U^- \to B^-$ by

$$F(u) := u \cdot \prod_{i\in I} \alpha_i^\vee(f_i(u)^{-1}). \tag{3.7}$$

Lemma 3.3. *The morphism $F : U^- \to B^-$ is a U-morphism.*

Proof. Let us verify that $\mathcal{T}$ satisfies (3.2). For $x \in U$ and $u \in U^-$ such that $xu \in Im(U^- \times T \times U \hookrightarrow G)$, let $u^- \in U^-$, $u^0 \in T$ and $u^+ \in U$ be the unique elements satisfying $u^-u^0u^+ = xu$, *i.e.*, $\pi^{--}(xu) = u^-$, $\pi^0(xu) = u^0$ and $\pi(xu) = u^+$. Since $\langle\ \ ,\ \ \rangle$ is a contragredient bilinear form and the fact that $g \cdot v^*_{\Lambda_i} = v^*_{\Lambda_i}$ for any $g \in U^-$, we have

$$\langle xu \cdot u_{\Lambda_i}, v^*_{\Lambda_i}\rangle = \langle u \cdot u_{\Lambda_i}, \omega(x) \cdot v^*_{\Lambda_i}\rangle = \langle u \cdot u_{\Lambda_i}, v^*_{\Lambda_i}\rangle. \tag{3.8}$$

On the other hand, since $g \cdot u_{\Lambda_i} = u_{\Lambda_i}$ for $g \in U$, we have

$$\begin{aligned}&\langle xu \cdot u_{\Lambda_i}, v^*_{\Lambda_i}\rangle = \langle \pi^-(xu)\pi^0(xu)\pi(xu) \cdot u_{\Lambda_i}, v^*_{\Lambda_i}\rangle \\ &= \langle \pi^-(xu)\pi^0(xu) \cdot u_{\Lambda_i}, v^*_{\Lambda_i}\rangle = \Lambda_i(\pi^0(xu))\langle \pi^-(xu) \cdot u_{\Lambda_i}, v^*_{\Lambda_i}\rangle,\end{aligned} \tag{3.9}$$

where $\Lambda_i \in X^*(T)$ such that $\Lambda_i(\alpha_j^\vee(c)) = c^{\delta_{i,j}}$. Hence, by (3.8), (3.9), we have

$$\begin{aligned} f_i(\pi^-(xu)) &= \langle \pi^-(xu) \cdot u_{\Lambda_i}, v^*_{\Lambda_i}\rangle = \Lambda_i(\pi^0(xu))\langle \pi^-(xu) \cdot u_{\Lambda_i}, v^*_{\Lambda_i}\rangle \\ &= \Lambda_i(\pi^0(xu))^{-1}\langle uu_{\Lambda_i}, v^*_{\Lambda_i}\rangle = \Lambda_i(\pi^0(xu))^{-1} f_i(u).\end{aligned}$$

By the formula

$$\prod_i \alpha_i^\vee(\Lambda_i(t)) = t, \quad (t \in T),$$

and the definitions of $\mathcal{T}$ and F, we obtained (3.2). □

Corollary 3.4. *Suppose that G is semi-simple. Then (U^-, F) is a unipotent crystal.*

As we have seen in 2.4, we can associate geometric crystal structure with the unipotent subgroup U^- since it has a unipotent crystal structure.

It is trivial that the function $\varphi_i : U^- \to \mathbb{C}$ is not identically zero. Thus, defining the morphisms $e_i : \mathbb{C}^\times \times U^- \to U^-$ and $\gamma_{U^-} : U^- \to T$ by

$$e_i(c, u) = e_i^c(u) := x_i\left(\frac{c-1}{\varphi_i(u)}\right)(u), \qquad \gamma_{U^-}(u) := \mathcal{T}(u), \qquad (u \in U^- \text{ and } c \in \mathbb{C}^\times), \tag{3.10}$$

It follows from Theorem 2.7:

Theorem 3.5. *If G is semi-simple, then the triplet $\chi_{U^-} := (U^-, \gamma_{U^-}, \{e_i\}_{i\in I})$ is a geometric crystal.*

4 $SL_{n+1}(\mathbb{C})$-case

We see the result of the previous section in the $SL_{n+1}(\mathbb{C})$-case more explicitly.

We identify unipotent subgroup U^- with the set of lower triangular matrices whose diagonal part is an identity matrix.

First, let us describe the morphism $\mathcal{T} : \mathcal{U}^- \to \mathcal{T}$. For $i \in I := \{1, \cdots, n\}$ and $u = (a_{ij})_{1 \le i,j \le n+1} \in U^-$, let $u^{(i)}$ be the submatrix with size i as:

$$u^{(i)} := (a_{i,j})_{n-i+2 \le i \le n+1, 1 \le j \le i},$$

i.e.,

$$u = \begin{pmatrix} 1 & & & \\ & 1 & & \Large 0 \\ \underbrace{\boxed{u^{(i)}}}_{i} & & \cdot & \\ & & & \cdot \\ & & & & 1 \end{pmatrix} \in U^-$$

and set $m_i(u) := \det(u^{(i)})$.

Let $V = \mathbb{C}^{n+1}$ be the $n+1$-dimensional vector space with the basis $\{u_1, u_2, \cdots, u_{n+1}\}$. We can identify V with the vector representation $L(\Lambda_1)$ of $\mathfrak{sl}_{n+1}$ by the standard way. Indeed, the explicit actions are given by:

$$e_i(u_j) = \delta_{i+1,j} u_{i-1} \quad f_i(u_j) = \delta_{i,j} u_{i+1},$$

where $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$(matrix unit), and we set $e_i(u_1) = 0$ and $f_i(u_{n+1}) = 0$ for all $i \in I$, which implies that u_1 is the highest weight vector and u_{n+1} is the lowest weight vector. Then we have the isomorphism between the fundamental representation $L(\Lambda_k)$ $(1 \le k \le n)$ and the k-th anti-symmetric tensor module $\bigwedge^k(V)$. Let us fix

$$u_{\Lambda_k} := u_1 \wedge u_2 \wedge \cdots \wedge u_k (\text{resp. } v_{\Lambda_k} := u_{n-k+2} \wedge u_{n-k+3} \wedge \cdots \wedge u_{n+1}) \tag{4.1}$$

the highest (resp. lowest)weight vector in $L(\Lambda_k) \cong \bigwedge^k(V)$. In this setting, we have

Lemma 4.1. *$f_i \equiv m_i$ on U^- for all $i = 1, \cdots, n$.*

Proof. For $g = (g_{ij}) \in U^-$, we have

$$g \cdot u_i = u_i + \sum_{i<j} g_{ji} u_j.$$

Let us see the coefficient of the vector $v_{\Lambda_k} := u_{n-k+2} \wedge u_{n-k+3} \wedge \cdots \wedge u_{n+1}$ in $g \cdot u_{\Lambda_k}$. We have

$$\begin{aligned} g \cdot u_{\Lambda_k} &= g \cdot u_1 \wedge g \cdot u_2 \wedge \cdots \wedge g \cdot u_k \\ &= (u_1 + \sum_{1<j} g_{j1} u_j) \wedge \cdots \wedge (u_k + \sum_{k<j} g_{jk} u_j). \end{aligned}$$

Thus, the coefficient of the vector $u_{j_1} \wedge \cdots \wedge u_{j_k}$ is $g_{j_1\,1} \cdots g_{j_k\,k}$. Hence, we obtain the coefficient of v_{Λ_k} as

$$\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) g_{n-\sigma(1)+2\,1} \cdots g_{n-\sigma(k)+2\,k} = m_i(u),$$

by using $v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k)} = \operatorname{sgn}(\sigma) v_1 \wedge v_2 \wedge \cdots \wedge v_k$. On the other hand, the coefficient of the lowest weight vector gives the function $f_i(u)$. Then we get the desired result. $\square$

Next, let us see the action of e_i^c on U^-. Indeed, the action of e_i^c is described simply by;

$$e_i^\alpha(u) = x_i(\frac{\varphi_i(u)}{\alpha-1})\cdot u\cdot x_i(\frac{1-\alpha}{\alpha\varphi_i(u)})\cdot\alpha_i^{-1}(\alpha).$$

Here, for the later purpose, we consider the following subset B^u of U^- and describe the action of e_i^α on it:

$$B^u := \left\{ Y(a) = \begin{array}{l} y_n(a_{1,n})y_{n-1}(a_{1,n-1})\cdots y_1(a_{1,1})\times \\ \times y_n(a_{2,n})\cdots\cdots y_2(a_{2,2})\times \\ \cdots\cdots\cdots \\ \times y_n(a_{n,n}) \end{array} : a_{i,j}\in\mathbb{C}^\times \right\} \subset U^-. \tag{4.2}$$

It is easy to see that B^u is an open dense subset in U^- and isomorphic to the algebraic torus $T_0 := (\mathbb{C}^\times)^{\frac{n(n+1)}{2}}$ by:

$$\begin{array}{rcl} T_0 = (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} & \xrightarrow{\sim} & B^u, \\ a = (a_{i,j})_{1\le i\le j\le n} & \mapsto & Y(a), \end{array}$$

which gives a birational isomorphism $\theta : T_0 = (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \to B^u \hookrightarrow U^-$.

Furthermore, we have

Lemma 4.2. $\varphi_i(Y(a)) = \sum_{k=1}^i a_{k,i}, \quad f_i(Y(a)) = \prod_{k=1}^i\prod_{j=k}^{n-i+k} a_{k,j}$.

Proof. Since for $u\in U^-$, $\varphi_i(u)$ is given as a $(i+1,i)$-entry and the $(i+1,i)$-entry of $Y(a)$ is $\sum_{k=1}^i a_{k,i}$, we obtained the first result.

For a word $\iota = i_1,\cdots,i_m$, we set $f_\iota := f_{i_1}\cdots f_{i_m}$. For a fixed reduced longest word $\iota_0 = n,\cdots,1,n\cdots,2,n\cdots,3,\cdots,n,n-1,n$, there exists the unique subword

$$\iota_i := \underbrace{n-i+1,\cdots,1}, \underbrace{n-i+2,\cdots,2}, \cdots, \underbrace{n-1,\cdots,i-1}, \underbrace{n,\cdots,i},$$

such that

$$f_{\iota_i}u_{\Lambda_i} = v_{\Lambda_i}, \tag{4.3}$$

where u_{Λ_i} (resp. v_{Λ_i}) is the highest (resp. lowest) weight vector of $L(\Lambda_i)$ as in (4.1). Since $f_i^2(L(\Lambda_k)) = \{0\}$, we have $y_i(a) = 1 + af_i$ on $L(\Lambda_k)$ and then

$$\begin{array}{rcl} Y(a) & = & (1+a_{1,n}f_n)\cdots(1+a_{1,1}f_1)\cdots(1+a_{n,n}f_n) \\ & = & \displaystyle\sum_{\iota:\text{subword of }\iota_0} a_\iota f_\iota, \end{array}$$

where a_ι is a coefficient of f_ι and $a_\iota f_\iota = 1$ if ι is empty. Hence by (4.3), we have

$$f_i(Y(a)) = \langle Y(a)u_{\Lambda_i}, v_{\Lambda_i}^*\rangle = a_{\iota_i} = \prod_{k=1}^i\prod_{j=k}^{n-i+k} a_{k,j}.$$

□

Let us see the rational action $e_i^\alpha : B^u \to B^u$.

Proposition 4.3. *We have* $e_i^\alpha Y((a_{k,j})_{1\le k\le j\le n}) = Y((a'_{k,j})_{1\le k\le j\le n})$,

$$a'_{k,j} := \begin{cases} C_k^{(i)}a_{k,i-1} & \text{if } j = i-1, \\ \frac{a_{k,i}}{C_{k-1}^{(i)}C_k^{(i)}} & \text{if } j = i, \\ C_{k-1}^{(i)}a_{k,i+1} & \text{if } j = i+1, \\ a_{k,j} & \text{otherwise}, \end{cases} \tag{4.4}$$

where
$$C_k^{(i)} := \frac{\alpha(a_{1,i}+\cdots+a_{k,i})+a_{k+1,i}+\cdots+a_{i,i}}{a_{1,i}+\cdots+a_{i,i}} \quad (1\le k\le i\le n).$$

Proof. We recall the formula:

$$x_i(a)y_j(b) = \begin{cases} y_i(\frac{b}{1+ab})\alpha_i^\vee(1+ab)x_i(\frac{a}{1+ab}) & \text{if } i=j \\ y_j(b)x_i(a) & \text{if } i\ne j \end{cases}, \tag{4.5}$$

$$\alpha_i^\vee(a)x_j(b) = x_j(a^{a_{ij}}b)\alpha_i^\vee(a), \quad \alpha_i^\vee(a)y_j(b) = y_j(a^{-a_{ij}}b)\alpha_i^\vee(a). \tag{4.6}$$

Using these formula repeatedly, we have

$$x_i(c)\cdot Y(a) = Y(a')\cdot\alpha_i^\vee(1+c\varphi_i(Y(a)))\cdot x_i(\frac{c}{1+c\varphi_i(Y(a))}),$$

where $c=(\alpha-1)/(a_{1,i}+\cdots+a_{i,i})$ and $\varphi_i(Y(a))=a_{1,i}+\cdots+a_{i,i}$. □

Now, we consider the following birational isomorphism:

$$\begin{array}{rccc} \xi: T_0=(\mathbb{C}^\times)^{\frac{n(n+1)}{2}} & \longrightarrow & T_0=(\mathbb{C}^\times)^{\frac{n(n+1)}{2}}, \\ (a_{i,j})_{1\le i\le j\le n} & \mapsto & (A_{i,j})_{1\le i\le j\le n} \end{array}$$

where
$$A_{i,j} := \frac{a_{i,j}a_{i-1,j-1}\cdots a_{1,j-i+1}}{a_{i-1,j}a_{i-2,j-1}\cdots a_{1,j-i+2}} \quad (1\le i\le j\le n).$$

The inverse morphism is given by

$$a_{i,j} := \frac{A_{i,j}A_{i-1,j}\cdots A_{1,j}}{A_{i-1,j-1}A_{i-2,j-1}\cdots A_{1,j-1}} \quad (1\le i\le j\le n). \tag{4.7}$$

We can describe explicitly

$$\begin{array}{rccc} \xi\circ e_i^\alpha\circ\xi^{-1}: (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} & \longrightarrow & (\mathbb{C}^\times)^{\frac{n(n+1)}{2}}, \\ (A_{k,j})_{1\le k\le j\le n} & \mapsto & (A'_{k,j})_{1\le k\le j\le n}, \end{array}$$

where
$$A'_{k,j} = \begin{cases} A_{k,j} & \text{if } j\ne i,i-1 \\ \alpha_k^{(i)}\cdot A_{k,i-1} & \text{if } j=i-1 \\ (\alpha_k^{(i)})^{-1}\cdot A_{k,i} & \text{if } j=i \end{cases}$$

$$\alpha_k^{(i)} = \frac{\alpha\displaystyle\sum_{1\le j\le k}\frac{\prod_{l=1}^j A_{l,i}}{\prod_{l=1}^{j-1}A_{l,i-1}} + \sum_{k<j\le i}\frac{\prod_{l=1}^j A_{l,i}}{\prod_{l=1}^{j-1}A_{l,i-1}}}{\alpha\displaystyle\sum_{1\le j\le k-1}\frac{\prod_{l=1}^j A_{l,i}}{\prod_{l=1}^{j-1}A_{l,i-1}} + \sum_{j=k}^{i}\frac{\prod_{l=1}^j A_{l,i}}{\prod_{l=1}^{j-1}A_{l,i-1}}}. \tag{4.8}$$

Set $\hat\theta := \theta\circ\xi^{-1}: (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \xrightarrow{\xi^{-1}} (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \xrightarrow{\theta} U^-$.

Theorem 4.4. *The morphism $\hat\theta$ gives a positive structure on the geometric crystal χ_{U^-}.*

Proof. The explicit form of

$$\begin{array}{rccc} \hat\theta^{-1}\circ e_i^\alpha\circ\hat\theta: (\mathbb{C}^\times)^{\frac{n(n+1)}{2}}\times\mathbb{C}^\times & \longrightarrow & (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \\ (A_{k,j})_{1\le k\le j\le n} & \mapsto & (A'_{k,j})_{1\le k\le j\le n} \end{array}$$

is given as (4.8), which is trivially positive. Then let us show the positivity of $\gamma_{U^-} \circ \hat{\theta}$. For $Y(a) \in B^u$, we have $\gamma_{U^-}(Y(a)) = \prod_i \alpha_i^\vee(f_i(Y(a))^{-1})$ and by Lemma 4.2 the explicit form of $f_i(Y(a))$ is given. Substituting (4.7) in it, we obtain

$$\gamma_{U^-} \circ \hat{\theta}((A_{k,j})_{1\leq k\leq j\leq n}) = \gamma_{U^-} \circ \theta \circ \xi^{-1}((A_{k,j})_{1\leq k\leq j\leq n}) = \prod_{i=1}^{n} \alpha_i^\vee(\prod_{\substack{1\leq k\leq i \\ i\leq j\leq n}} A_{k,j})^{-1}, \quad (4.9)$$

which implies that $\gamma \circ \hat{\theta}$ is positive. $\square$

5 Tropicalization of Geometric Crystals on U^- and generalized Young Tableaux

5.1 Crystal structure on Young tableaux

Let us recall the crystal structure on Young tableaux where the terminology "Young tableaux" means "semi-standard tableaux" in [7]. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, set

$$B(\lambda) = \{ \text{ Young tableau of shape } \lambda \text{ with contents } 1, 2, \cdots, n+1\},$$

which gives an A_n-crystal of irreducible highest weight $U_q(\mathfrak{sl}_{n+1})$-module $V(\lambda)$[7].

In order to describe the action of $\tilde{e}_i^\beta(b)$ $(\beta \geq 0)$ explicitly, let us recall how to construct $B(\lambda)$ following [7],[5].

Let $V_\square := V(\Lambda_1)$ be the vector representation of $U_q(\mathfrak{sl}_{n+1})$, which is the irreducible highest weight module with the highest weight Λ_1 and let

$$B_\square := \left\{\boxed{1}, \boxed{2}, \cdots, \boxed{n+1}\right\}$$

be the crystal of $V_\square$. The explicit actions of $\tilde{e}_i$ and $\tilde{f}_i$ are given as follows([7]):

$$\tilde{e}_i\boxed{j} = \delta_{i+1,j}\boxed{i-1}, \qquad \tilde{f}_i\boxed{j} = \delta_{i,j}\boxed{i+1}.$$

We realize $B(\lambda)$ by embedding into $B_\square^N$ $(N = |\lambda|)$, which follows the way of embedding $V(\lambda) \hookrightarrow V_\square^N$. In [7], the "*Japanese reading*" is introduced, which gives the embedding by reading entries in a Young tableau column by column. But here we take so-called "*arabic reading*" [5], which gives the embedding by reading entries in a Young tableau row by row from right to left since it matches what we do below.

Example 5.1. (*i*) *Japanese reading*

$$\begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & & \\ \hline g & & & \\ \hline\end{array} = \left(\boxed{d}\right) \otimes \left(\boxed{c}\right) \otimes \left(\boxed{b} \otimes \boxed{f}\right) \otimes \left(\boxed{a} \otimes \boxed{e} \otimes \boxed{g}\right) \in B_\square^{\otimes 7}$$

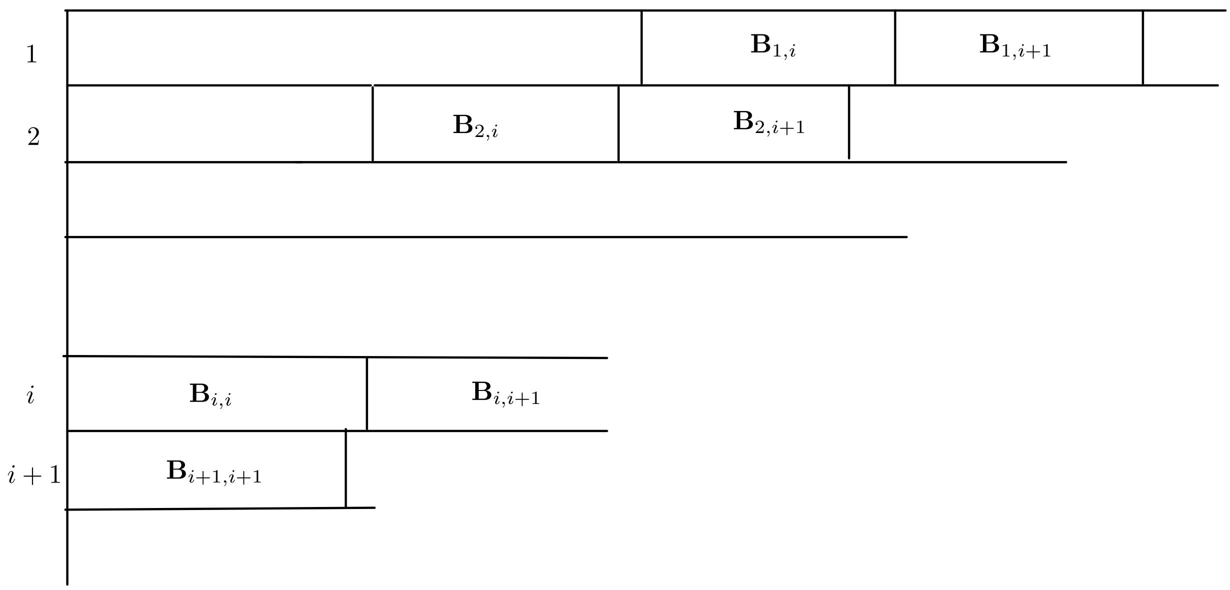
(*ii*) *Arabic reading*

$$\begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & & \\ \hline g & & & \\ \hline\end{array} = \left(\boxed{d} \otimes \boxed{c} \otimes \boxed{b} \otimes \boxed{a}\right) \otimes \left(\boxed{f} \otimes \boxed{e}\right) \otimes \left(\boxed{g}\right) \in B_\square^{\otimes 7}$$

The description of the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B(\lambda)$ in [7] is as follows: Let $\{(+),(-)\}$ (resp. $\{(0)\}$)be the crystal of the irreducible $U_q(\mathfrak{sl}_2)$-module $V_{\square}$ (resp. $V(0)$). If we consider the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on a tensor product $B_{\square}^N$, we can identify ([7],2.1.),

$$\boxed{i} = (+), \quad \boxed{i+1} = (-), \quad \boxed{j} = (0) \ (j \neq i, i+1). \tag{5.1}$$

Let $b \in B(\lambda)$ be in the following form:



(5.2)

where $B_{i,j} := \sharp\{j \text{ in the } i\text{-th row }\}$. If we consider the actions of $\tilde{e}_i$ and $\tilde{f}_i$, by the "arabic reading" and (5.1) we can identify:

$$b = v_1 \otimes \cdots \otimes v_{i+1},$$

where

$$v_k := (-)^{\otimes B_{k,i+1}} \otimes (+)^{\otimes B_{k,i}} \quad (1 \leq k \leq i), \quad v_{i+1} = (-)^{\otimes B_{i+1,i+1}} \tag{5.3}$$

For any $i \in I$ and $\beta \in \mathbb{Z}_{\geq 0}$ there exist unique $\beta_k^{(i)} \in \mathbb{Z}_{\geq 0}$ $(1 \leq k \leq i+1)$ such that

$$\tilde{e}_i^{\beta}(v_1 \otimes \cdots \otimes v_{i+1}) = \tilde{e}_i^{\beta_1^{(i)}}(v_1) \otimes \cdots \otimes \tilde{e}_i^{\beta_{i+1}^{(i)}}(v_{i+1}), \tag{5.4}$$

and $\beta = \sum_{1 \leq k \leq i+1} \beta_k^{(i)}$. Note that on each component, we have

$$\tilde{e}_i^{\beta_k^{(i)}}(v_k) := (-)^{\otimes(B_{k,i+1} - \beta_k^{(i)})} \otimes (+)^{\otimes(B_{k,i} + \beta_k^{(i)})}.$$

Let us see the explicit form of $\beta_k^{(i)}$, in order to describe the action of $\tilde{e}_i^{\beta}$ on b. For the purpose, we prepare the following formula:

Lemma 5.2 ([5]). *Let B_1, $B_2, \cdots, B_l$ be crystals. For $v_k \in B_k$ and $i \in I$, set $b_k := \varepsilon_i(v_k) - \sum_{1 \leq j < k} \langle h_i, wt(v_j) \rangle$. Then, we have*

$$\tilde{e}_i^c(v_1 \otimes \cdots \otimes v_l) = \tilde{e}_i^{c_1}(v_1) \otimes \cdots \otimes \tilde{e}_i^{c_l}(v_l),$$

where

$$c_k = \max(c + \max_{1 \leq j \leq k}(b_j), \max_{k < j \leq l}(b_j)) - \max(c + \max_{1 \leq j < k}(b_j), \max_{k \leq j \leq l}(b_j)). \tag{5.5}$$

Applying this lemma to (5.4), we have

Proposition 5.3. *Under the setting (5.3) and (5.4),*

$$
\begin{aligned}
\beta_k^{(i)} &= \max\left(\beta + \max_{1\le j\le k}\left(\sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i}\right), \max_{k<j\le i}\left(\sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i}\right)\right) \\
&\quad -\max\left(\beta + \max_{1\le j< k}\left(\sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i}\right), \max_{k\le j\le i}\left(\sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i}\right)\right) \quad (1\le k\le i), \\
\beta_{i+1}^{(i)} &= 0
\end{aligned}
$$

Proof. By (5.3), we have $\varepsilon_i(v_k) = B_{k,i+1}$, $\langle h_i, wt(v_j)\rangle = B_{j,i} - B_{j,i+1}$. Applying this to Lemma 5.2, we obtain

$$
\begin{aligned}
\beta_k^{(i)} = \max&\left(\beta + \max_{1\le j\le k}\left(\sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i}\right), \max_{k<j\le i+1}\left(\sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i}\right)\right) \\
-\max&\left(\beta + \max_{1\le j< k}\left(\sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i}\right), \max_{k\le j\le i+1}\left(\sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i}\right)\right). \qquad (5.6)
\end{aligned}
$$

Since $B_{i+1,i+1} \le B_{i,i}$, we have

$$
\sum_{l=1}^{i} B_{l,i+1} - \sum_{l=1}^{i-1} B_{l,i} \ge \sum_{l=1}^{i+1} B_{l,i+1} - \sum_{l=1}^{i} B_{l,i}.
$$

Hence, we can neglect $j = i+1$ in the formula (5.6). □

Remark. The formula $\beta_k^{(i)}$ does not depend on $B_{i,i}$ or $B_{i+1,i+1}$.

5.2 Generalized Young tableaux and its crystal structure

Let $b \in B(\lambda)$ be a Young tableau as in (5.2). The $B_{i,j}$'s have several constraints, *e.g.,*

$$
B_{i,j} \ge 0, \qquad \sum_{i\le j\le k} B_{i,j} \ge \sum_{i+1\le j\le k+1} B_{i+1,j},
$$

which come from the conditions for being Young tableaux.

Now, forgetting such constraints on $B_{i,j}$'s, we obtain a free $\mathbb{Z}$-lattice $B^\sharp$:

$$
B^\sharp := \{(B_{i,j})_{1\le i<j\le n+1} | B_{i,j} \in \mathbb{Z}\} (= \mathbb{Z}^{\frac{1}{2}n(n+1)}),
$$

Now, we define the action of $\tilde{e}_i^\beta$ $(\beta \ge 0)$ on $B^\sharp$ by

$$
\tilde{e}_i^\beta((B_{k,j})_{1\le k<j\le n+1}) = ((B_{k,j} + \beta_{k,j})_{1\le k<j\le n+1}), \qquad \beta_{k,j} := \begin{cases} \beta_k^{(i)} & \text{if } j = i, \\ -\beta_k^{(i)} & \text{if } j = i+1, \\ 0 & \text{otherwise} \end{cases} \tag{5.7}
$$

Here note that in the definition of $B^\sharp$, $B_{i,i}$'s do not appear since the formula $\beta_k^{(i)}$ does not depend on $B_{i,i}$'s as mentioned in the remark of the last subsection.

The explicit action of the Kashiwara operator $\tilde{e}_i$ (resp. $\tilde{f}_i$) on $B^\sharp$ is given by (5.7) taking $\beta = 1$ (resp. $\beta = -1$). Indeed, the crystal structure of $B^\sharp$ is described as follows:

For $v = (B_{i,j}) \in B^\sharp$ set

$$b_k^{(i)}(v) := \sum_{1\le l\le k} B_{l,i+1} - \sum_{1\le l\le k} B_{l,i},$$

$$\begin{cases} \varepsilon_i(v) := \max_{1\le k\le i}\{b_k^{(i)}(v)\}, \\ wt(v) := -\sum_{i=1}^{n}(\sum_{\substack{1\le k\le i \\ i+1\le j\le n+1}} B_{k,j})\alpha_i, \\ \varphi_i(v) := \langle h_i, wt(v)\rangle + \varepsilon_i(v), \end{cases} \tag{5.8}$$

$$m_i = m_i(v) := \min\{k | 1 \le k \le i,\ b_k^{(i)}(v) = \varepsilon_i(v)\}$$
$$M_i = M_i(v) := \max\{k | 1 \le k \le i,\ b_k^{(i)}(v) = \varepsilon_i(v)\}.$$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $v = (B_{i,j})$ are given by

$$\tilde{f}_i : \begin{cases} B_{k,j} \longrightarrow B_{k,i} & \text{if } (k,j) \ne (M_i, i), (M_i, i+1) \\ B_{M_i,i} \longrightarrow B_{M_i,i} - 1 & \text{if } (k,j) = (M_i, i) \\ B_{M_i,i+1} \to B_{M_i,i+1} + 1 & \text{if } (k,j) = (M_i, i+1) \end{cases} \tag{5.9}$$

$$\tilde{e}_i : \begin{cases} B_{k,j} \longrightarrow B_{k,i} & \text{if } (k,j) \ne (m_i, i), (m_i, i+1), \\ B_{m_i,i} \longrightarrow B_{m_i,i} + 1 & \text{if } (k,j) = (m_i, i) \\ B_{m_i,i+1} \to B_{m_i,i+1} - 1 & \text{if } (k,j) = (m_i, i+1) \end{cases} \tag{5.10}$$

Theorem 5.4. *By the setting (5.8), (5.9) and (5.10), we obtain a free crystal $B^\sharp$.*

Proof. It suffices to check the axioms (2.6)–(2.10) in Definition 2.8 and the bijectivity of $\tilde{e}_i$ or $\tilde{f}_i$. Indeed, (2.6)–(2.8) are trivial from (5.8), (5.9) and (5.10). The assumption of (2.10) never occurs. Thus, we may show that $\tilde{e}_i\tilde{f}_i = \mathrm{id} = \tilde{f}_i\tilde{e}_i$. For $v = (B_{i,j})$, set $p := M_i(v)$, which implies

$$b_1^{(i)}(v), \cdots, b_{p-1}^{(i)}(v) \le b_p^{(i)}(v) > b_{p+1}^{(i)}(v), \cdots, b_i^{(i)}(v).$$

By the definition of $b_k^{(i)}$ and the action of $\tilde{f}_i$, we have

$$b_k^{(i)}(v) = \begin{cases} b_k^{(i)}(v) & 1 \le k < p, \\ b_p^{(i)}(v) + 1 & k = p, \\ b_k^{(i)}(v) + 2 & p < k \le i. \end{cases}$$

Thus, we have

$$b_1^{(i)}(\tilde{f}_i v), \cdots, b_{p-1}^{(i)}(\tilde{f}_i v) < b_p^{(i)}(\tilde{f}_i v) \ge b_{p+1}^{(i)}(\tilde{f}_i v), \cdots, b_i^{(i)}(\tilde{f}_i v),$$

which means $M_i(v) = p = m_i(\tilde{f}_i(v))$. Similarly, we have $m_i(v) = M_i(\tilde{e}_i v)$. It follows from these that $\tilde{e}_i\tilde{f}_i = \mathrm{id}_{B^\sharp} = \tilde{f}_i\tilde{e}_i$ and then we get (2.9) and the bijectivity of $\tilde{e}_i$ and $\tilde{f}_i$. □

Remark. It is unknown whether the crystal graph of $B^\sharp$ is connected or not.

5.3 Tropicalization of $B^\sharp$

Let us see that a tropicalization of the crystal $B^\sharp$ is the geometric crystal on the unipotent subgroup $U^- \subset SL_{n+1}(\mathbb{C})$ treated in Sect.4.

Applying the following correpondence to (4.8) and (5.6), and (4.9) and (5.8)

$$\begin{aligned} x\cdot y &\longleftrightarrow x+y \\ x/y &\longleftrightarrow x-y \\ x+y &\longleftrightarrow \max(x,y) \\ i &\longleftrightarrow i+1 \end{aligned}$$

we obtain $\alpha_k^{(i)} \leftrightarrow \beta_k^{(i)}$ and then

$$\mathcal{UD}_{\hat\theta,T_0}(e_i^c)=\tilde e_i^c, \qquad \mathcal{UD}_{\hat\theta,T_0}(\gamma)=wt, \tag{5.11}$$

(where $T_0 := (\mathbb{C}^\times)^{\frac{n(n+1)}{2}}$), which implies the following theorem:

Theorem 5.5. *We have $\mathcal{UD}_{\hat\theta,T'}(\chi_{U^-})=(B^\sharp, wt, \{\tilde e_i\}_{i\in I})$, i.e., the geometric crystal χ_{U^-} on $U^-\subset SL_{n+1}(\mathbb{C})$ defined in Sect 4 is a tropicalization of the crystal $B^\sharp=(B^\sharp, wt, \{\tilde e_i\}_{i\in I})$.*

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